Journal of Applied Mechanics and Technical Physics, Vol. 45, No. 2, pp. 222–229, 2004

UNSTEADY MOTION OF A GAS IN A STRIP

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UDC 533.06

An exact solution is constructed, which describes a gas glow in a strip between a rectilinear source and sink. With time, the strip turns and expands. In the case of consistent boundary conditions, the flow in the strip is continuous. If the consistency constraints are violated, a shock wave is formed inside the strip.

Key words: *polytropic gas, invariant solution, shock wave, projective transformation.*

Introduction. The group classification of equations of motion of an inviscid heat-non-conducting gas was first performed by Ovsyannikov [1]. He found the following noteworthy fact: in the case of a polytropic gas with a special polytropic exponent $\gamma = (n+2)/n$ (n is the number of spatial variables), the gas-dynamic equations (GDE) admit a projective transformation (PT) in addition to the "conjecture" Galilean group and extensions. Later, this transformation was independently found by Nikol'skii [2, 3]. The specific feature of PT is that it does not follow from the "physical" properties of space uniformity, Galilean principle of relativity, etc. At the same time, as was shown in [4], PT generates new conservation laws in GDE.

One possible PT application for obtaining exact GDE solutions is based on the fact that the solution is again transformed to the solution under the action of the admitted transformation. Using this method for generation of new solutions from the known solutions, Nikol'skii obtained unsteady GDE solutions generated by a constant solution and also by a flow of the type of a spherical source or sink. Based on the solution obtained by PT from a simple Riemann wave, Nikol'skii found a solution of the problem of discontinuity decay, where the uniformly expanding or compressing gas borders on a vacuum region at the initial time. With the help of the known Sedov's solution [5], the problem of a point explosion in a uniformly expanding or compressing medium was considered. The same research direction includes the publication [6], where it was shown that it is possible to obtain new unsteady solutions with functional arbitrariness from known steady GDE solutions on the basis of PT and also Munk and Prim's transformation [7] available in the steady case.

There are some examples of using PT for constructing exact invariant solutions. On the basis of PT, Khabirov [8] obtained an exact solution of equations of two-dimensional gas dynamics. Within the shallow-water model, this solution is treated as spreading and rotation of a liquid ring with an arbitrary initial cross section. Another invariant solution constructed with the use of PT is considered in [9]. An invariant submodel on a one-dimensional subalgebra containing a projective operator is constructed there. The solution invariant with respect to the full group of rotations is considered for this submodel. Strong discontinuities on such solutions in one-, two-, and three-dimensional cases are described. Classes of PT-generated invariant solutions are identified in [10]. All PT-generated invariant and regular, partly invariant GDE submodels ($\dim = 2 + 1$ and $\gamma = 2$) are listed and preliminary examined in [11]. One more model constructed on the basis of PT is described in [12]. The level lines in this submodel are rays emanating from the origin. The special feature of the solution is its invariance with respect to discrete symmetry: rotation around the origin by a certain fixed angle.

One invariant solution obtained in [11] is described in detail in the present paper. The level lines in the examined solution are straight lines that form an orthogonal grid on the plane at each time instant. The exact GDE solution is determined only in a certain strip rotating and expanding with time. The strip boundaries are a gas source and a gas sink of certain intensity. If the source and sink intensities are consistent, the flow in the strip

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Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 45, No. 2, pp. 90–98, March–April, 2004. Original article submitted October 27, 2003.

is continuous. In the case of inconsistent data at the strip boundaries, a shock wave arises inside the strip. The entire solution (including the case with the shock wave) is described by finite formulas.

1. Preliminary Information. The gas-dynamic equations for a polytropic gas with a polytropic exponent $\gamma = 2$ are written in standard notation of velocity $\boldsymbol{u} = (u, v)$, density ρ, pressure p, and entropy S:

$$\begin{gathered} \rho D\boldsymbol{u} + \nabla p = 0, \qquad D\rho + \rho \operatorname{div} \boldsymbol{u} = 0, \qquad Dp + 2p \operatorname{div} \boldsymbol{u} = 0, \\ p = S\rho^2, \qquad D = \partial_t + \boldsymbol{u} \cdot \nabla, \qquad \nabla = (\partial_x, \, \partial_y). \end{gathered} \tag{1.1}$$

The functions $\boldsymbol{u}$, p, ρ, and S depend on the spatial coordinates $\boldsymbol{x} = (x, y)$ and time t.

In the case $S = \text{const}$, Eqs. (1.1) coincide with the model of motion of a thin layer of a potential liquid in the gravity field above a flat bottom (shallow water model). The height of the liquid layer is a hydrodynamic analog of density. In what follows, we use this analogy for a clearer interpretation of the solutions obtained.

Equations (1.1) admit the 10-dimensional Lie group G_{10} of transformations. It consists of translations along the coordinate axes and in time, Galilean translations, rotation, three extensions, and projective transformation. The optimal system of subgroups ΘG_{10} and the list of invariant and regular, partly invariant submodels for system (1.1) can be found in [10, 11]. One invariant submodel of Eqs. (1.1) generated by the subgroup containing a projective transformation is identified and considered in detail below.

2. Invariant Submodel of Rank 2. We choose a one-dimensional subalgebra generated by the operator

$$H_1 = (t^2+1)\partial_t + (-y+tx)\partial_x + (x+ty)\partial_y + (x-tu-v)\partial_u + (y+u-tv)\partial_v + (\alpha-2t)\rho\partial_\rho + (\alpha-4t)p\partial_p.$$

The parameter α can take arbitrary real values. The finite transformation corresponding to the operator H_1 consists in simultaneous rotation in the plane Oxy, translation in time, extensions p and ρ, and projective transformation. The representation of the solution of rank 2 is written as

$$\begin{gathered} u = \frac{tU(\lambda,\mu) + V(\lambda,\mu) + tx - y}{t^2+1}, \qquad v = \frac{-U(\lambda,\mu) + tV(\lambda,\mu) + ty + x}{t^2+1}, \\[2ex] p = \frac{P(\lambda,\mu)}{(t^2+1)^2}\, \mathrm{e}^{\alpha \arctan t}, \qquad \rho = \frac{R(\lambda,\mu)}{t^2+1}\, \mathrm{e}^{\alpha \arctan t}, \qquad c^2 = \frac{2p}{\rho} = \frac{Z(\lambda,\mu)}{t^2+1}, \\[2ex] S = S(\lambda,\mu)\, \mathrm{e}^{-\alpha \arctan t}, \qquad \lambda = \frac{tx-y}{t^2+1}, \qquad \mu = \frac{ty+x}{t^2+1}. \end{gathered} \tag{2.1}$$

Here, the invariant functions U, V, P, R, and S depending on the invariant variables λ and μ are determined by the following system:

$$\begin{gathered} UU_\lambda + VU_\mu + \frac{1}{R}\, P_\lambda = 2V, \qquad UV_\lambda + VV_\mu + \frac{1}{R}\, P_\mu = -2U, \\[2ex] UR_\lambda + VR_\mu + R(U_\lambda + V_\mu) = -\alpha R, \qquad US_\lambda + VS_\mu = \alpha S, \qquad P = SR^2. \end{gathered} \tag{2.2}$$

System (2.2) differs from equations defining two-dimensional steady motions of the gas by the presence of a nonzero right side. We find the role of the coordinates λ and μ with respect to the physical coordinates t, x, and y. Note, the lines $\lambda = \text{const}$ and $\mu = \text{const}$ at each fixed time form an orthogonal grid on the plane Oxy. At the initial time $t = 0$, this grid coincides with the coordinate lines $x = \text{const}$, $y = \text{const}$. With increasing time, the grid $O\lambda\mu$ turns anticlockwise around the center O so that the angle of rotation reaches $\pi/2$ as $t \to \infty$. Simultaneously, uniform extension of the grid $O\lambda\mu$ occurs. As $t \to \infty$, each point $\lambda = \lambda_0$, $\mu = \mu_0$ tends to infinity on the physical plane Oxy. Thus, the "steady" pattern of the flow in terms of invariant variables corresponds to spreading with simultaneous anticlockwise rotation on the physical plane.

The trajectory of a particle starting at $t = 0$ from the point (x_0, y_0) is described by the following equations in invariant variables:

$$\frac{d\lambda}{dt} = \frac{U}{t^2+1}, \quad \frac{d\mu}{dt} = \frac{V}{t^2+1}, \qquad \lambda(0) = -y_0, \quad \mu(0) = x_0. \tag{2.3}$$

System (2.2) admits the algebra L_4 consisting of translations along λ and μ, rotation, and extension. For $\alpha = 0$, we can introduce the stream function $\psi(\lambda, \mu)$ by the equalities

$$UR = \psi_\mu, \qquad VR = -\psi_\lambda. \tag{2.4}$$

The following first integrals of system (2.2) are valid:

$$U^2 + V^2 + 2Z = F(\psi), \qquad S = S(\psi). \tag{2.5}$$

Here, F and S are arbitrary functions.

For flows with discontinuities of the shock-wave (SW) type described by system (2.2), we have to write the Hugoniot conditions in terms of invariant variables. We assume that the front of a strong discontinuity is defined by the formula $h(\lambda, \mu) = \text{const}$. We denote the component of velocity $\boldsymbol{u}$ normal to the front as u_n and the front velocity along the normal as D_n. Then, the relative velocity of gas motion is

$$u_n - D_n = \frac{1}{\sqrt{t^2+1}} \frac{Uh_\lambda + Vh_\mu}{\sqrt{h_\lambda^2 + h_\mu^2}} = \frac{1}{\sqrt{t^2+1}} U_n. \tag{2.6}$$

The Hugoniot conditions in terms of invariants (the constant α is assumed to remain unchanged when passing through the SW) are

$$\begin{gathered} [RU_n] = 0, \qquad [RU_n^2 + P] = 0, \qquad [U_n^2 + 4P/R] = 0, \\ [S] > 0, \qquad [Uh_\mu - Vh_\lambda] = 0. \end{gathered} \tag{2.7}$$

The submodel of Eqs. (2.2) invariant with respect to translation along μ is considered in detail below.

3. Invariant Submodel of Rank 1. We consider the invariant solution of gas-dynamic equations (1.1) with respect to the two-dimensional subalgebra $L_2 = \{H_1, H_2\}$ with the following operator H_2:

$$H_2 = \partial_x + t\partial_y + \partial_v.$$

In the initial variables, the operator H_2 defines the transformation of simultaneous translation along the axis Ox and Galilean translation along the Oy axis. In invariant variables of submodel (2.2), the operator H_2 corresponds to the translation along μ. Thus, the representation of the submodel solution coincides with (2.1) where the invariant functions U, V, P, R, S, and Z depend only on one invariant variable λ. The submodel equations are found from (2.2) in the following form:

$$\begin{gathered} UU' + P'/R = 2V, \qquad UV' = -2U, \\ RU' + UR' = -\alpha R, \qquad US' = \alpha S, \qquad P = SR^2, \qquad Z = 2P/R \end{gathered} \tag{3.1}$$

(the prime denotes the derivative with respect to λ). In studying the submodel, we have to distinguish several cases.

(a) $\alpha = 0$ and $U \equiv 0$. The solution is described by the formulas

$$U = 0, \qquad V = S'R/2 + SR', \qquad R = R(\lambda), \quad S = S(\lambda) \tag{3.2}$$

with arbitrary functions $R(\lambda)$ and $S(\lambda)$. In invariant variables, this solution is an analog of the shear solution. For $S = 1/2$, we obtain a solution for the shallow-water equations. The function $R(\lambda)$ defines the profile of the cross section $x = \text{const}$ of the free surface of the liquid at $t = 0$. At the initial time, the free surface has the form of a cylinder with the generatrix parallel to the Ox axis and the guiding line $R(-y)$. With increasing time, the liquid bounded by this surface turns anticlockwise and spreads.

If the function $R(\lambda)$ has a continuous derivative, the entire solution is continuous on the plane. If the derivative $R'(\lambda)$ has a discontinuity of the first kind at a certain $\lambda = \lambda_*$, we obtain a motion with a contact discontinuity. Indeed, the normal component of velocity U and the liquid depth R are continuous at the discontinuity line. The velocity component V tangential to the straight line $\lambda = \lambda_*$ has a discontinuity of the first kind. Solution (3.2) cannot adjoin the invariant shock wave with the front equation $\lambda = \text{const}$ because the relative gas velocity U equals zero (the gas does not flow through the wave front).

(b) $\alpha = 0$ and $U \neq 0$. With accuracy to insignificant constants, the solution of Eqs. (3.1) is described by the set of the first integrals

$$\begin{gathered} U^2 + 2Z + 4\lambda^2 = D^2, \qquad UZ = m, \qquad V = -2\lambda, \qquad S = S_0, \\ D, m, S_0 = \text{const}, \qquad D^2 > 3m^{2/3}. \end{gathered} \tag{3.3}$$

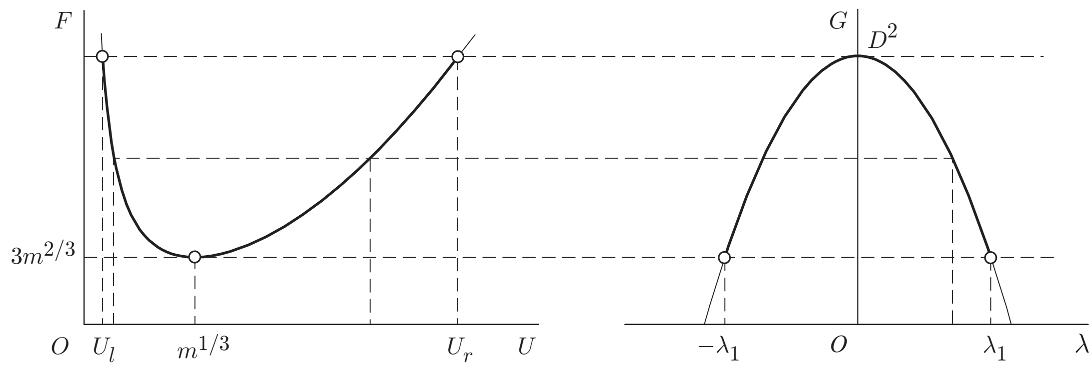

Fig. 1. Reconstruction of the dependence $U(\lambda)$ by comparing the plots of the functions $F(U)$ and $G(\lambda)$: $F(U) = G(\lambda)$.

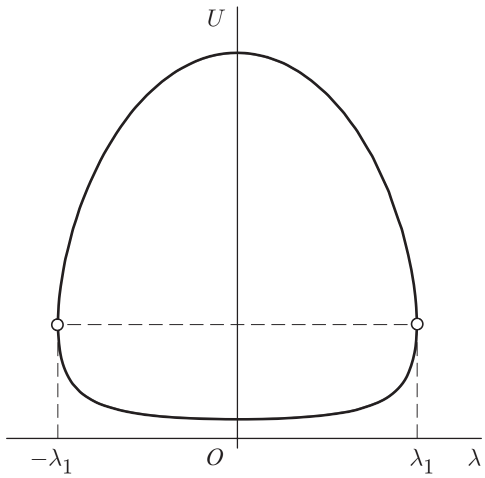

Fig. 2. Two-valued dependence $U(\lambda)$. The upper (lower) branch of the function corresponds to gas motion with $U^2 > Z$ ($U^2 < Z$).

Note, system (2.2) admits the transformation (involution)

$$\lambda \to -\lambda, \qquad \mu \to -\mu, \qquad U \to -U, \qquad V \to -V, \tag{3.4}$$

which allows us to assume that $U > 0$ and $m > 0$ (by definition, $Z \geqslant 0$). We introduce the notation

$$F(U) = U^2 + 2m/U, \qquad G(\lambda) = D^2 - \lambda^2.$$

The dependence $U(\lambda)$ is two-valued (Fig. 1). The plot of the function $U(\lambda)$ is shown in Fig. 2. The solution is determined on the finite interval $\lambda \in (-\lambda_1, \lambda_1)$ with $\lambda_1 = \sqrt{D^2 - 3m^{2/3}}$. The straight lines $\lambda = \lambda_1$ are coupled $C_\pm$ characteristics of gas-dynamic equations (1.1). The derivative $U'(\lambda)$ vanishes on these lines. The function U takes the values from an interval separated from zero. The upper (lower) branch of the solution corresponds to gas motions where the relative velocity (2.6) is greater (smaller) than the local velocity of sound. At the boundary points $\lambda = \lambda_1$, these velocities coincide.

(c) $\alpha \neq 0$. Integration reduces system (3.1) to one first-order ordinary differential equation (ODE) and the set of the first integrals:

$$UZ = m, \qquad V = -2\lambda, \qquad S = S_0 \exp\Big(\alpha \int U^{-1}(\lambda)\, d\lambda\Big); \tag{3.5}$$

$$U' = -\frac{8\lambda U^2 - \alpha m}{2(U^3 - m)}. \tag{3.6}$$

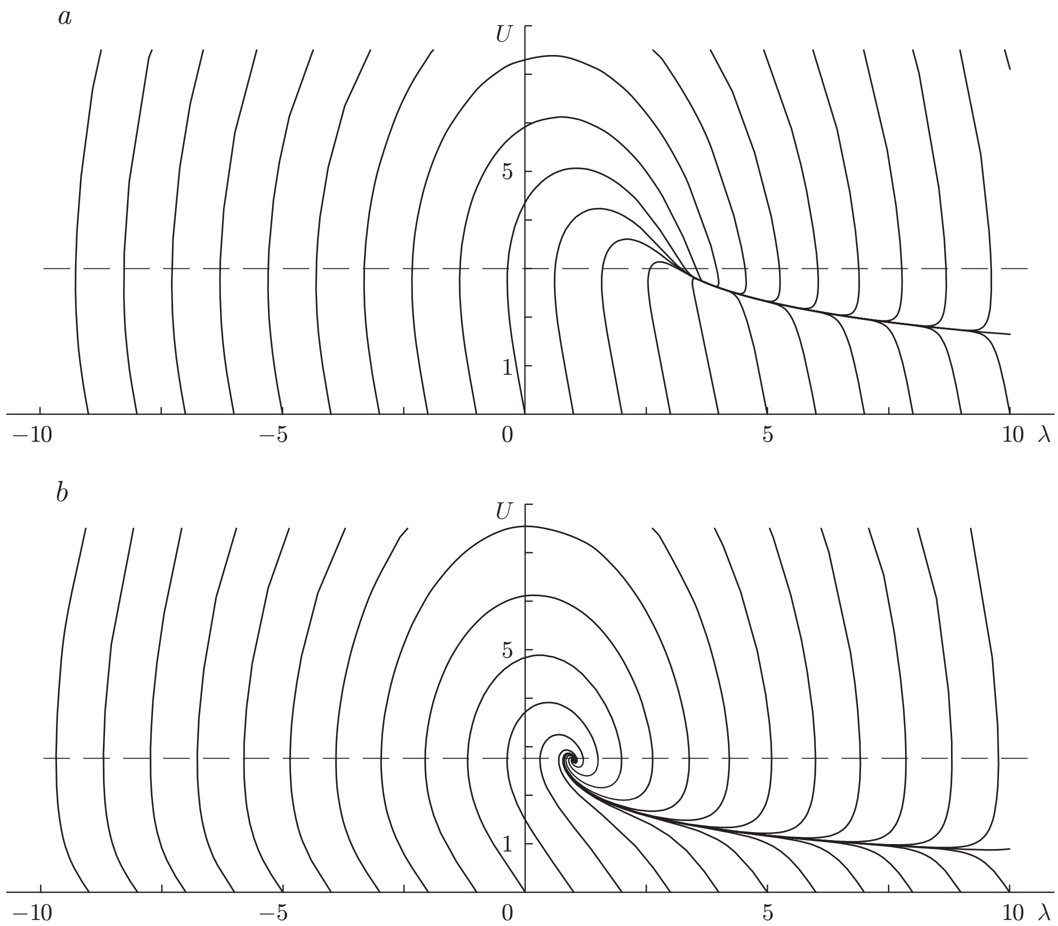

Fig. 3. Integral curves of Eq. (3.6): (a) the singular point is a node ($m = 10$ and $\alpha = 4\sqrt{3} + 4$); (b) the singular point is a focus ($m = 10$ and $\alpha = 4\sqrt{3} - 4$).

Equation (3.6) for finite λ and U has one singular point $\lambda = \alpha m^{1/3}/8$, $U = m^{1/3}$, which is a focus for $|\alpha| < 4\sqrt{3}$, a node for $|\alpha| > 4\sqrt{3}$, and a degenerate node for $|\alpha| = 4\sqrt{3}$. By virtue of involution (3.4) supplemented by the transformation $\alpha \to -\alpha$, the pattern of integral curves on the plane (λ, U) for $\alpha < 0$ is obtained from the pattern with an identical absolute value $\alpha > 0$ by reflection about the axis $\lambda = 0$. Typical patterns of integral curves are shown in Fig. 3, where the dashed line is the straight line $U = m^{1/3}$, where $U'(\lambda) \to \infty$. The unique solution $U(\lambda)$ of Eq. (3.6) is also determined only on a finite interval of variation of λ.

Note, in both cases ($\alpha = 0$ and $\alpha \neq 0$), the physical meaning of the constant m is determined as the flow rate of the gas passing through the cross section $\lambda = \text{const}$.

4. Shock Wave. We prove that it is possible to adjoin the solutions of submodel (3.1) through the shock wave. Let the invariant shock wave be described by the equation $\lambda = \lambda_*$. The gas velocity relative to the SW front, according to (2.6), is $U/\sqrt{t^2+1}$, i.e., we should use $U_n = U$ in relations (2.7). Using the invariant velocity of sound Z, we write the Hugoniot relations (2.7) as

$$[RU] = 0, \qquad [U^2 + 2Z] = 0, \qquad [U + Z/(2U)] = 0, \qquad [V] = 0, \qquad [S] > 0. \tag{4.1}$$

The first relation in (4.1) is equivalent to $[ZU/S] = 0$. Hence, using the first integral of $ZU = m$, we obtain

$$S_2 = S_1 \frac{Z_2 U_2}{Z_1 U_1} = S_1 \frac{m_2}{m_1}. \tag{4.2}$$

The law of the increase in entropy on the shock wave yields $m_2 > m_1$. For solutions (3.3) and (3.5), (3.6), Eq. (4.2) holds with an appropriate choice of the constants S_0 in solutions ahead of and behind the shock wave. The relation $[V] = 0$ is automatically satisfied by virtue of the special dependence $V(\lambda)$ determined by formulas (3.3) and (3.5). It is convenient to analyze the remaining relations in (4.1) separately for solutions with $\alpha = 0$ and $\alpha \neq 0$.

For $\alpha = 0$, it is convenient to choose the constant m and the relative velocity U as variables that characterize the state ahead of and behind the shock wave. It follows from formulas (3.3) and the second Hugoniot condition in (4.1) that $[D] = 0$. For convenience, we perform an extension transformation such that $D = \sqrt{3}$. Then, the flow parameters ahead of the shock wave m_1 and U_1 are arbitrarily chosen from the domain

$$\Omega = \{U_1^2 + 2m_1/U_1 < 3, \quad m_1 < 1, \quad m_1 < U_1^3\}. \tag{4.3}$$

The remaining, third relation in (4.1) determines the value of the constant m_2 behind the SW in the form

$$m_2 = m_1 + 4(U_1^3 - m_1)^3/(27U_1^6). \tag{4.4}$$

We have to show that the conditions of existence of a solution in the form (3.3) are satisfied for the constant m_2 calculated in accordance with (4.4), i.e., $m_2 < 1$. This condition turns out to be always satisfied; it is verified by finding a conventional extremum of function (4.4) in domain (4.3). Note, by virtue of the Zemplén theorem, the relative velocity U should be greater than the velocity of sound ahead of the shock wave $U_1^2 > Z_1$ and smaller than the velocity of sound behind the wave $U_2^2 < Z_2$. Hence, it follows that a jump from the upper branch of the function $U(\lambda)$ to the lower branch (with a different constant m) occurs when passing through the shock wave.

For $\alpha \neq 0$, the solution is determined by formulas (3.5) and (3.6). It is convenient to use the values of the relative velocity U ahead of and behind the shock wave to describe the shock transition. From the second and third relations in (4.1), using the integral $ZU = m$ and making obvious transformations, we obtain the expressions for the constants m_1 and m_2:

$$m_1 = U_1^2(3U_2 - U_1)/2, \qquad m_2 = U_2^2(3U_1 - U_2)/2. \tag{4.5}$$

By virtue of the Zemplén theorem, we have $m_1 = U_1 Z_1 < U_1^3$. Substituting the expressions for the constant m_1 from (4.5), we obtain the following restriction: $U_2 < U_1$. There are no other restrictions on the choice of the limiting value of the function U on the SW sides. The solution of the form (3.5), (3.6) containing the SW is constructed as follows:

— the constants U_1 and U_2 ($U_2 < U_1$) are chosen arbitrarily;

— the values of m_1 and m_2 are calculated by formulas (4.5);

— the SW position $\lambda = \lambda_*$ is chosen arbitrarily;

— the solution ahead of the SW $\lambda < \lambda_*$ is found by solving Eq. (3.6) with the initial data $U(\lambda_*) = U_1$;

— the solution behind the SW $\lambda > \lambda_*$ is found from the Cauchy problem for Eq. (3.6) with the initial data $U(\lambda_*) = U_2$;

— the values of the remaining functions are reconstructed from the known $U(\lambda)$ in accordance with formulas (3.5).

Note, by virtue of the Zemplén theorem, the jump on the SW occurs from the domain $U^2 > Z$ (integral curves above the dashed straight line in Fig. 3) to the domain $U^2 < Z$ (integral curves below the dashed straight line in Fig. 3).

5. Description of Motion. The particle trajectories are calculated by formulas (2.3). On the plane of invariants (λ, μ), the invariant streamlines are found by the equation

$$\frac{d\mu}{d\lambda} = -\frac{2\lambda}{U(\lambda)}. \tag{5.1}$$

It follows from this equation that the function $\mu(\lambda)$ is increasing for $\lambda < 0$ and decreasing for $\lambda > 0$. The limiting lines $\lambda = \pm\lambda_1$ should be considered as a source and sink of the gas. The normal velocity of gas motion on these limiting lines equals the velocity of sound $U^2 = Z$, and the tangential component of velocity is $V = \mp 2\lambda_1$. The flow rate of the source and sink is determined by the constant m. If the source and sink have identical flow rates, the flow between the limiting lines is continuous. If the constants m for the source and sink are not consistent ($m_2 > m_1$), a shock wave described by the formulas given above appears in the solution. The qualitative pattern of the flow [by the example of solution (3.3)] is illustrated in Figs. 4 and 5, where the solid line shows the invariant streamlines for a flow with $U^2 > Z$. The dashed line in Fig. 4 indicates the invariant streamlines calculated for $U^2 < Z$. The dashed line in Fig. 5 refers to the SW front; the states ahead of the SW front and behind it are indicated by 1 and 2, respectively. On the physical plane, the whole pattern turns anticlockwise and uniformly spreads from the center O.

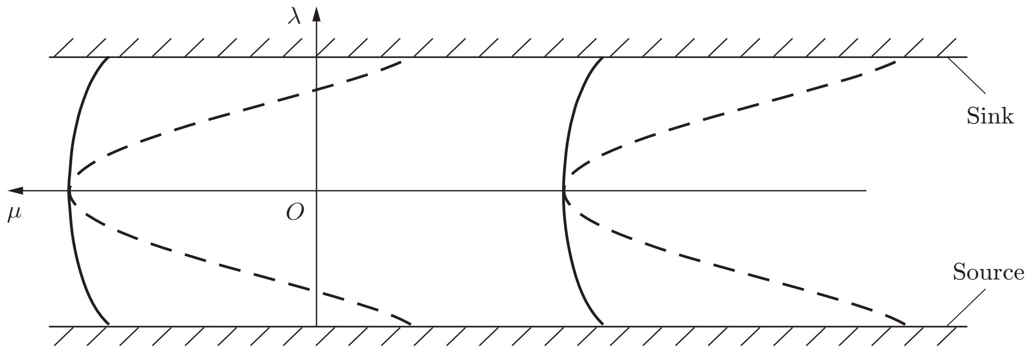

Fig. 4. Continuous motion of the gas.

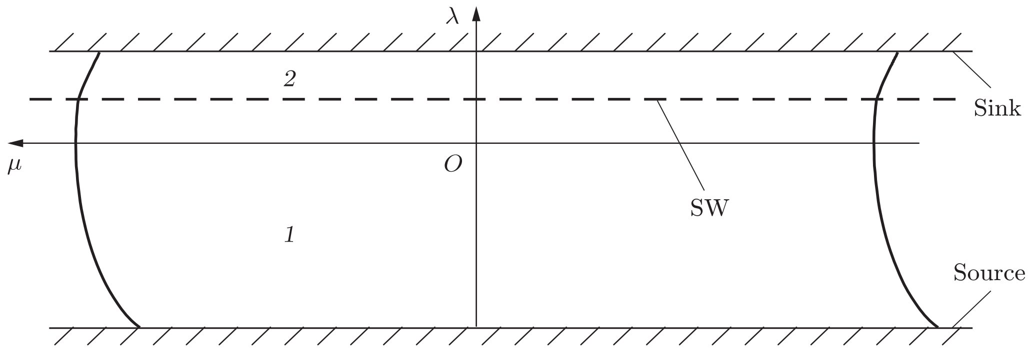

Fig. 5. Motion of the gas with a shock wave.

Conclusions. An invariant submodel with straight level lines is constructed in the present work. The submodel describes unsteady two-dimensional motions of a polytropic gas. In the case of constant entropy, the same submodel defines the motion of a thin layer of an ideal liquid above a flat horizontal bottom. There are three types of solutions. The solution of the first type defines a rotational shear motion. Within the framework of the shallow-water model, this motion is treated as spreading of a liquid "ridge," which has the form of a cylinder with an arbitrary initial cross section. Solutions of the second and third type define gas motions in a strip between a linear source and a linear sink parallel to it, which turn and move away from each other with time. If the flow rates of the source and sink are consistent, the flow between them is continuous. In the case of inconsistent flow rates, a shock wave is formed between the source and sink. In the second-type solution, the entropy is constant and described by finite formulas. The solution of the third type defines a motion with variable entropy: its determination reduces to solving one first-order ODE.

This work was supported by the Russian Foundation for Basic Research (Grant No. 02-01-00550) and within the framework of the Young Scientists' Project (No. 18) of the Siberian Division of the Russian Academy of Sciences.

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